

Analysis Qualifying Exam  
August, 2021

Passing levels:

MS: You must do, in total, at least 4 problems completely correctly, or 3 completely correctly with substantial progress on 2 others. You are free to choose the problems from either or both sections.

PhD: You must do: a) at least 2 completely correctly from **each** of the two sections; and b) at least 6 completely correctly in total, or 5 completely correctly with substantial progress on 2 others.

Notation:  $\mathbf{R}$  means the real numbers and  $\mathbf{C}$  means the complex numbers. If  $z \in \mathbf{C}$  then  $\operatorname{Re} z$  means  $z$ 's real part and  $\operatorname{Im} z$  is its imaginary part. If  $f : X \rightarrow Y$  and  $A \subseteq X$  then  $f[A] := \{f(x) : x \in A\}$ .

**Section I: Real analysis.**

1. Let  $(X; M; \mu)$  be a measure space and  $f \in L^1(\mathbf{R}; M; \mu)$ . Show that, for all  $\epsilon > 0$ , there is a  $\delta > 0$  so that, if  $E \in M$  and  $\mu(E) < \delta$ , then  $\int_E |f| d\mu < \epsilon$ .

2. Let  $f \in L^\infty(\mathbf{R}; L; m)$  (the usual Lebesgue space on the line). For  $t > 0$  define

$$G(t) := \int_{\mathbf{R}} e^{-t|x|} f(x) dx;$$

where the integral is assumed to be a Lebesgue integral. Show that  $G$  is defined and continuous on all of  $(0; \infty)$ . You may assume standard calculus facts about the exponential function.

3. Let  $(X; d)$  be a metric space. Show that, if  $\{x_n\}$  and  $\{y_n\}$  are two Cauchy sequences in  $X$ , then

$$\lim_{n \rightarrow \infty} d(x_n; y_n)$$

exists as a real number, where the limit is taken with respect to the usual  $j$ -based metric.

4. Let  $(X; d)$  be a connected metric space, and let  $f : X \rightarrow \mathbf{R}$  have the property that, for every  $p \in X$ , there is an  $r > 0$  such that  $f$  is constant on  $B(p; r) := \{x \in X : d(x; p) < r\}$ .

Show that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

exists and equals 0.

7. Consider  $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  defined by  $f(x; y) := (x^2 + 2xy; xy + y^2)$ . Show that, if  $(a; b) \neq (0; 0)$ , there is an open  $U \subset \mathbf{R}^2$  with  $(a; b) \in U$  such that  $f$  is one-to-one on  $U$ ,  $f[U]$  is open, and there is a differentiable  $g : f[U] \rightarrow U$  such that  $g(f(x; y)) = (x; y)$  for all  $(x; y) \in U$ .

8. Let  $(X; d)$  be a compact metric space, where “compact” means “every open cover of  $X$  has a finite subcover”. Show that every infinite sequence  $f_n \in X$  has a subsequence converging to some  $p \in X$ .

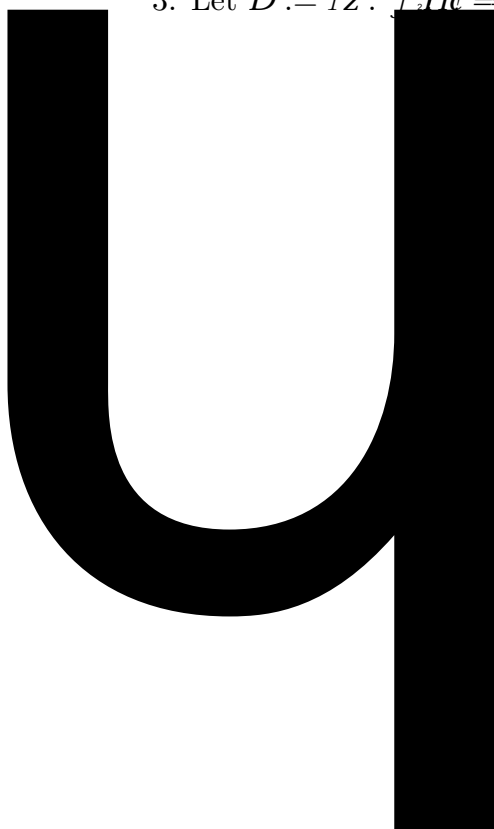
### Section II: Complex analysis.

1. Find an analytic bijection  $f : \{z : \operatorname{Im} z > 0\} \rightarrow \{z : \operatorname{Im} z < 0\}$ . Write your bijection as a sequence of compositions of analytic bijections, with sketches of the intermediate domains.

2. Use residues to show that

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(1+x^2)^2} dx = \frac{3}{2e^2}.$$

3. Let  $\bar{D} := \{z : |z| \leq 1\} \cup \{z : |z| = 1\}$



on the annulus  $fz : 2 < |z| < 5$ .

7. Let  $f : \mathbf{C} \rightarrow \mathbf{C}$  be entire, and suppose there are positive numbers  $a$  and  $b$  such that  $|f(z)| > a$  whenever  $|z| > b$ . Show that  $f$  is a polynomial.

8. Let  $p(z)$  and  $q(z)$  be two non-trivial polynomials with different degrees. Show that there is no entire  $f : \mathbf{C} \rightarrow \mathbf{C}$  such that

$$|p(z)| = |f(z)| = |q(z)|$$

for all  $z \in \mathbf{C}$ .