ANALYSIS QUALIFYING EXAM
18 May 2020

Passing at the PhD level is accomplished by solving at least two problems in each section while either solving a total of 6 problems or solving 5 problems and making signi cant progress on two others.

R means the real numbers, C is the complex numbers, and Q is the rational numbers. If $E \subset X$ is a set then $\chi_E : X \to$ R means

$$\chi_E(\ ) := \begin{cases} 1 & \text{if} \quad E; \\ 0 & \text{if} \quad X \quad E. \end{cases}$$

Complex Analysis.

1. Let f be analytic on C 0 and suppose that, for all C 0 ,

$$f(\ ) \le \quad ^{-1=3} + \quad ^{3=4}.$$

Show that f is constant.

2. Use residues or a substitution to show that

$$\int_{-\infty}^{\infty} \frac{\log}{1 + \quad ^2} \, d \quad = 0$$

and then use that fact plus residues to show

$$\int_{-\infty}^{\infty} \frac{(\log \quad)^2}{1 + \quad ^2} \, d \quad = \frac{\pi^3}{4}.$$

3. Let $f, g :$ C $\to$ C be entire and suppose that

$$f(\ ) \le g(\ ) + f(\ )$$

for all C. Show that f, g is a linearly dependent set: λ_1, λ_2 C, not both equal to 0, such that $\lambda_1 f(\ ) + \lambda_2 g(\ ) = 0$ for all C. (Hint: Divide)

4. Let $\Omega :=$ C $((- \quad , -1] \quad [1, \quad))$. Find an analytic bijection $f : \Omega \to \quad : \quad > 0$. Express your f as a sequence of compositions, sketching the intermediate domains.

5. Let $f :$ C $\to$ C be entire and suppose that, for each C

7. Define

$$f(\ ) := \frac{\ + 1}{^2 + \ \ - 2}.$$

Find a Laurent expansion for f, of the form

$$\sum_{-\infty}^{\infty} c_n(\ \ + 1)^n,$$

which converges to f in $\quad \mathbb{C} : 1 < \ \ + 1 \ < 2$.

Real Analysis.

8. Let (M, d) be a metric space. Show that, if $A, B \subset M$ are closed and $A \quad B = \ $, then there exist open sets U, V such that $A \subset U$, $B \subset V$, and $U \quad V = \ $. (Hint: U and V are unions of open balls. How do you choose the radius of each ball?)

9. Let $(X, \quad, \mu)$ be a measure space and let $\phi : X \to [0, \quad]$ be measurable. For E define $\lambda(E) := \int_E \phi \, d\mu = \int \phi \chi_E \, d\mu$. Use standard limit theorems to show that λ defines a measure on and that, if $f : X \to [0, \quad]$ is measurable, then

$$\int f \, d\lambda = \int f \phi \, d\mu.$$

You may use without proof the fact that there exists a sequence $\ \psi_n \ {}_1^{\infty}$ of non-negative measurable simple functions such that $\psi_n(\ ) \leq \psi_{n+1}(\ )$ for all and , and $\psi_n(\ ) \to f(\ )$ pointwise as $\ \to \ $.

10. Use standard limit theorems from measure theory and facts from calculus (about the integrals of exponentials, etc.) to show that

$$\int_0^{\infty} \frac{e^{-x}}{1 - e^{-x}} \, d \quad = \sum_1^{\infty} \frac{1}{k^2},$$

where `d ' means integration with respect to Lebesgue measure. (Hint: Use a geometric series.)

11. Enumerate the rationals $\mathbb{Q} := \ \ _1, \ _2, \ _3, \ldots \ $, and define

$$f(\ ) := \sum_1^{\infty} \frac{2^k}{k^2} \chi_{(q_k - 2^{-k}; q_k + 2^{-k})}(\ ).$$

Show that

$$\int_{\mathbf{R}} f(\ ) \, d \quad <$$

(where `d ' means integration with respect to Lebesgue measure), but that, for every $\ > 1$ and every non-empty open $U \subset \mathbb{R}$,

$$\int_U f(\ )^p \, d \quad = \quad .$$

12. Let $f : [a,b] \to \mathbf{R}$ be continuous, with $f(a) \le f(b)$, and suppose that f has no local maximum or minimum on (a,b). Show that f is non-decreasing on all of $[a,b]$: $\quad , y$ $[a,b](\quad < y \qquad f(\quad) \le f(y))$. (Hint: First show that, if $\qquad [a,b]$, then $f(a) \le f(\quad) \le f(b)$.)

13. Let $f : [a,b] \to \mathbf{R}$ be continuous, with $[a,b] \subset \mathbf{R}$ and $a < b$. Use standard facts about continuous functions and the Riemann integral to show that

$$\lim_{n\to\infty} \left(\int_a^b \ f(\quad) \ ^n \ d \quad \right)^{1=n}$$

exists and equals $\max_{a;b} \ f(\quad) \ .$

14. Show that, if $\ a_k \ {}_1^{\infty}$ is any sequence of non-negative numbers and $0 < \quad < \quad < \qquad$ then

$$\left(\sum_1^{\infty} a_k^q \right)^1$$