(where $(-1)^0 := 1$), justifying your answer using appropriate limit theorems from measure theory and basic calculus facts about exponentials and power series.

7. Let $f : [0,1] \to \mathbb{C}$ be continuous in the usual sense, and define

$$M := \max_{x \in [0,1]} |f(x)|,$$

which we know exists because $[0,1]$ is compact. Show that

$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n \, dx \right)^{1/n}$$

exists and equals M.

Section II: Complex analysis.

1. Use residues to show that

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{e^x + e^{-x}} \, dx = \frac{\pi}{e^{\pi} + e^{-\pi}}.$$

2. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and not a polynomial. Show that, for all $a \in \mathbb{C}$ and $0 < r < \infty$,

$$f[\mathbb{C} \setminus \Delta(a;r)]$$

is dense in $\mathbb{C}$, where $\Delta(a;r) := \{z \in \mathbb{C} : |a - z| < r\}$.

3. Set $D := \{z \in \mathbb{C} : |z| < 1\}$. Write down, as a sequence of compositions of analytic bijections, an analytic bijection $f : \Omega \to H$, where

$$\Omega := D \setminus ((-1, -1/2] \cup [1/2, 1))$$
$$H := \{z \in \mathbb{C} : \Re z > 0\}.$$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be entire, with $f(0) = 0$. Show that

$$\varphi(z) := \sum_{n=1}^{\infty} f(z/n^2)$$

converges for all $z \in \mathbb{C}$ and defines an entire function.

5. Let $\Omega \subset \mathbb{C}$ be a connected open set, and $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ both analytic. Suppose that $\overline{f}g$ is analytic, where $\overline{f}$ means f's complex conjugate. Show that either f is constant or g is identically 0 on Ω.

6. Let

$$f(z) := \frac{3z - 7}{z^2 - 5z + 6}.$$

Find the Laurent expansion for f which is valid in $\mathrm{ann}(1, 1, 2) := \{z \in \mathbb{C} : 1 < |z - 1| < 2\}$.

7. State some form of the Maximum Principle for functions f analytic on a connected open $\Omega \subseteq \mathbb{C}$. Use it (with other standard facts about analytic functions) to prove the following: If $\Omega \subseteq \mathbb{C}$ is connected and open, $f : \Omega \to \mathbb{C}$ is analytic, and there is some $a \in \Omega$ such that

$$\Re f(a) \leq \Re f(z)$$

for all $z \in \Omega$, then f is constant. We are using $\Re w$ to mean the real part of a complex number w.