COMBINATORICS QUALIFYING EXAM
May 2020

This exam has three sections: A, B and C.

PhD Pass: Four numbered questions solved completely, with at least one from each section.

MS Pass: Substantial progress on three questions, in any section.

You have three hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

Section A

Question 1

(a) State both Hall's Theorem and König's Theorem, which both concern matchings in bipartite graphs.

(b) You will do one step of the proof of König's Theorem. Let G be a bipartite graph with bipartition $V(G) = A \cup B$, and let M be a maximum matching in G. From every edge in M, we add one of its endpoints to a set U: its endpoint in B if an alternating path ends there, and its endpoint in A otherwise. Explain why, if an alternating path P ends in a vertex $b \in B$, then we must have $b \in U$.

(c) Derive Hall's Theorem from König's Theorem. (You only need to prove the sufficiency of Hall's Condition.)

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Question 2

(a) State Vizing's Theorem regarding the edge-chromatic number of graphs.

(b) Let H be a connected, 3-regular graph. Prove that if H is Hamiltonian, then H is properly 3-edge colorable.

(c) We propose the following greedy algorithm to find a proper coloring of the edges of a graph G. For a fixed ordering on the edges $e_1, e_2, \ldots, e_m$, where $m = |E(G)|$, consider the edges in turn and color each edge with the first available color (i.e. the smallest positive integer not already used to color an edge that shares an endpoint with e_i among $e_1, \ldots, e_{i-1}$).
Does this algorithm always find an optimal coloring? Prove or show a counter-example.

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Section B

Question 1

Consider the recurrence

$$g_n = \begin{cases} 0; & \text{for } n < 0; \\ 1; & \text{for } n = 0; \\ g_{n-1} + 2g_{n-2} + \cdots + ng_0; & \text{for } n > 0: \end{cases}$$

(a) Find a functional equation for $G(x) = \sum_n g_n x^n$.

(b) Conjecture an expression for g_n in terms of the Fibonacci numbers.

(c) Prove your conjecture from part (b).

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Question 2

(a) Row insert the word $[8; 2; 2]$ into the tableaux
$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$.

(b) The Pieri rule is a special case of the Littlewood-Richardson rule used for computing the product of a complete homogeneous symmetric function h_n with a Schur function s_λ. Using either the general Littlewood-Richardson rule or the simplified Pieri rule, compute the Schur expansion of the product $h_3 s_{32}$.

(c) The Giambelli rule is analogous to the Pieri rule, but is used for computing the product of an elementary symmetric function e_m by a Schur function s_λ. Explain how to derive the Giambelli rule from the Pieri rule using the properties of the ω involution.

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Section C

Question 1

Let M be a matroid on ground set E, and K be a field with cardinality $|K|$.

(a) Let A be a matrix with entries from K whose columns give a respresentation of M. Describe how to use A to produce a representation of the dual matroid M^* over K.

(b) Let U_n^r be the uniform matroid of rank r on n elements. Prove that U_n^2 is representable over K if and only if $|K| \geq n - 1$.

(c) Over which fields is U_n^{n-2} representable?

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Question 2

Let $G = (E; V)$ be a graph and Z_G be the graphical zonotope associated to G. For this question we will fix $G = C_4$ the cycle of length 4.

(a) How many vertices does Z_{C_4} have?

(b) How many integer points lie in the interior of Z_{C_4}?

(c) How many maximal parallelepipeds are there in a fine tiling, e.g. a paving, of Z_{C_4}?

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