5. Let x and y be independent indeterminates over the field C of complex numbers, and let $R = \mathbb{C}[x, y]/(x^2 - y, y^2 - x)$.

   (a) Explain why R is a finite dimensional vector space over C, and find its dimension.

   (b) Prove that R is isomorphic to $\mathbb{C}[x]/(x^4 - x)$.

   (c) Show that R is (ring) isomorphic to the direct product of four copies of C.

6. Let t be an indeterminate over Q. Classify all finitely generated modules over the ring $\mathbb{Q}[t]/(t^9)$.

7. (a) Find all possible rational canonical forms for a 4×4 matrix A over Q that satisfies $A^6 = I$ (where I is the identity matrix).

   (b) Find all possible rational canonical forms for a 4×4 matrix A over $\mathbb{F}_2$ that satisfies $A^6 = I$.

Section C.

8. Let K be the splitting field of $(x^2 - 3)(x^3 - 5)$ over Q.

   (a) Find the degree of K over Q.

   (b) Find the isomorphism type of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

   (c) Find, with justification, all subfields F of K such that $[F$:

Alternate Questions

3. Show that Z_3 and S_3 are the only finite groups with exactly three conjugacy classes. [You may quote without proof the fact that Z_2 is the only finite group with exactly two conjugacy classes.] (too easy?)
3. Find all finite groups G such that $\mathrm{Aut}(G)$ is cyclic.

2. If both A and B are abelian and $A \cap B = 1$, prove that AB is abelian.

   (c) If both A and B are solvable, prove that AB is solvable.

4. Let R be a commutative ring with 1.

   (a) Prove that if R has only finitely many elements, then every nonzero element of R is either a unit or a zero divisor.

6. Describe all irreducible (simple) R-modules where $R = \mathbb{Z}[x]$ is the integer polynomial ring in the variable x.
10. Suppose there were a subfield, F, of $\mathbb{C}$ such that $[\mathbb{C} : F] = 5$. You may assume that $\mathbb{C}$ is algebraically closed to do the following:

    (a) Show that F contains a primitive 5^{th} root of unity.

    (b) Explain why $\mathbb{C} = F(\sqrt[5]{\alpha})$ for some $\alpha \in F$? (You may quote results about solvable extensions.)