# ALGEBRA PH.D. QUALIFYING EXAM

January 12, 2018

Three hours

5. Let $R$ be a nonzero ring with 1 and let $M$ be an $R$-module containing an $R$-submodule $N$.

   (a) Prove that if $N$ and $M/N$ are both finitely generated, then $M$ is also finitely generated.

   (b) Assume that $M$ is the direct sum

$$M = \bigoplus_{n=1}^{\infty} M_n \qquad \text{where each } M_n \cong R.$$

   Prove that $M$ is not a finitely generated $R$-module. (Recall that the above direct sum is the set of tuples $(r_1, r_2, r_3, \ldots)$ where only finitely many $r_n \ne 0$ in each tuple.)

6. Find the number of similarity classes of $10 \times 10$ matrices $A$ with entries from $\mathbb{Q}$ satisfying $A^{10} = I$ but $A^i \ne I$ for $1 \le i \le 9$, where $I$ is the identity matrix. (You do not need to exhibit representatives of the classes.)

## Section C.

7. Let $f(x) = x^4 - 8x^2 - 1 \in \mathbb{Q}[x]$, let $\alpha$ be the real positive root of $f(x)$, let $\beta$ be a nonreal root of $f(x)$ in $\mathbb{C}$, and let $K$ be the splitting field of $f(x)$ in $\mathbb{C}$.

   (a) Describe $\alpha$ and $\beta$ in terms of radicals involving integers, and deduce that $K = \mathbb{Q}(\alpha, \beta)$.

   (b) Show that $[\mathbb{Q}(\alpha^2) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^2)] = 2$. Deduce from this that $f(x)$ is irreducible over $\mathbb{Q}$.

   (c) Show that $[K : \mathbb{Q}] = 8$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_8$.

8. Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree $n$ and let $K$ be the splitting field of $f(x)$ in $\mathbb{C}$. Assume $G = \mathrm{Gal}(K/\mathbb{Q})$ is abelian.

   (a) Prove that $[K : \mathbb{Q}] = n$ and that $K = \mathbb{Q}(\alpha)$ for every root $\alpha$ of $f(x)$.

   (b) Prove that $G$ acts regularly on the set of roots of $f(x)$. (A group acts regularly on a set if it is transitive and the stabilizer of any point is the identity.)

   (c) Prove that either all the roots of $f(x)$ are real numbers or none of its roots are real.

9. Let $p$ be a prime and let $n \in \mathbb{Z}^+$ with $(p, n) = 1$. Let $K$ be the splitting field of the polynomial $x^n - 1$ over the finite field $\mathbb{F}_p$ of order $p$. Prove that $[K : \mathbb{F}_p] = d$, where $d$ is the order of $p$ in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$.