Algebra Qualifying Exam | Fall 2019

You have three hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Section B

4. Let R be a ring with 1 and let M be a simple left R-module (this means that M has no left R-submodules other than 0 and M).

   (a) If $\varphi : M \to M$ is a non-trivial R-module homomorphism (i.e. an endomorphism), show that φ is an isomorphism.

   (b) Show that if $m \in M$ with $m \neq 0$, then $M = Rm$.

   (c) Show that there is a left R-module isomorphism $M \cong R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of R.

5. Let $R = \mathbb{R}[x]/(x^4 - 1)$, so R is a commutative ring with 1.

   (a) Show that all ideals of R are principal.

   (b) Find a generator for each maximal ideal of R.

   (c) For each maximal ideal $\mathfrak{m}$, describe an isomorphism from $R/\mathfrak{m}$ to either $\mathbb{R}$ or $\mathbb{C}$.

6. Let R be a commutative ring with 1 which is a subring of the commutative ring S. Let P be a prime ideal of S.

   (a) Show that $P \cap R$ is a prime ideal of R.

   (b) Show that $P[x]$ is a prime ideal of $S[x]$.

   (c) Show that $P[x]$ is not a maximal ideal of $S[x]$.

Section C

7. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $= \sqrt{2} \quad \sqrt{3}$.

   (a) Show that $[L(\sqrt{}) : L] = 2$ and $[L(\sqrt{}) : \mathbb{Q}] = 8$.

   (b) Find the minimal polynomial of $\sqrt{}$ over $\mathbb{Q}$.

   (c) Show that $L(\sqrt{})$ is not Galois over $\mathbb{Q}$.

8. Let be the real, positive fourth root of 5, and let $i = \sqrt{1} \in \mathbb{C}$. Let $K = \mathbb{Q}(\ ; i)$.

   (a) Explain why $K/\mathbb{Q}$ is a Galois extension with Galois group dihedral of order 8.

   (b) Find the largest abelian extension of $\mathbb{Q}$ in K (i.e. the unique largest sub eld of K that is Galois over $\mathbb{Q}$ with abelian Ga[illegible]