REAL AND COMPLEX ANALYSIS PHD QUALIFYING EXAM

May 20, 2008

The test has two sections, covering real and complex analysis. In order to pass, you must do at least 2 problems from each section **completely correctly**, and you must do a total of 6 problems completely correctly, or 5 completely correctly with substantial progress on 2 others. Some problems have more than one part (e.g., problem 1 in Section I consists of 1a), 1b), and 1c)).

I. REAL ANALYSIS.

1a) State what it means for a function $f : [a; b]$

You may use without proof the standard tests (ratio, root, etc.) for computing radii of convergence, as well as the values of limits taught in elementary calculus: e.g., that $n^{-1} \log n \to 0$ as $n \to \infty$.

5. Consider the two functions $f_1(x, y, z) \equiv x^2 - 2x + y^2$ and $f_2(x, y, z) \equiv x^2 + y^2 + z^2 - 4$, each mapping from $\mathbb{R}^3$ into $\mathbb{R}$. (Note that f_1 does *not* depend on z.) Define

$$\Sigma \equiv \{(x, y, z) \in \mathbb{R}^3 : f_1(x, y, z) = f_2(x, y, z) = 0\}.$$

At what points (x', y', z') on Σ does the Implicit Function Theorem *not* guarantee the existence of an open neighborhood U of z' and differentiable functions $g : U \mapsto \mathbb{R}$ and $h : U \mapsto \mathbb{R}$ such that $(g(z'), h(z'), z') \in \Sigma$ for all $z' \in U$? You do not need to sketch Σ, but it will probably help you to do so.

6. Find, using the appropriate limit theorem or theorems,

$$\lim_{n \to \infty} \int_0^{\infty} \frac{x^{1/n}}{(e^x + \frac{x}{n})} \, dm(x).$$

II. COMPLEX ANALYSIS.

In this section, D always denotes the set $\{z \in \mathbb{C} : |z| < 1\}$.

1. Use residues to show that, for all $a \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2} \, dx = \pi e^{-a}.$$

2. How many zeroes, counting multiplicities, does $f(z) \equiv 3z^2 + e^z$ have inside D?

3a) State and sketch a proof of Morera's Theorem.

3b) Let $f : D \mapsto \mathbb{C}$ be continuous on all of D, and suppose that f is analytic on $D \setminus \{0\}$. Use Morera's Theorem and Cauchy's Theorem (don't prove Cauchy's Theorem) to show that f is analytic on all of D.

4. Consider the function $u(x, y) \equiv e^x \cos y + x^3 - 3xy^2$, which maps from $\mathbb{R}^2$ into $\mathbb{R}$. Show that u is harmonic on $\mathbb{R}^2$, and find a harmonic $v : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$f(z) \equiv f(x + iy) = u(x, y) + iv(x, y)$$

is analytic on all of $\mathbb{C}$.

5. Let

$$f(z) = \frac{z}{z^2 - z - 2}.$$

This function has a Laurent series expansion of the form

$$f(z) = \quad {}^{1}$$