REAL ANALYSIS PHD QUALIFYING EXAM

September 27, 2008

The test has 10 questions. To pass you must do 7 problems completely correctly or do 6 completely correctly and show substantial progress on 2 others.

1. Let $\{a_k\}_1^\infty$ be a sequence of real numbers such that $\sum_1^\infty |a_k| < \infty$. Set $s = \sum_1^\infty a_k$. (You don't need to show that s exists.) Show that the value of this sum does not depend on the order in which the a_k's are added. In other words, show that if $\sigma : \mathbb{N} \mapsto \mathbb{N}$ is a one-to-one mapping of $\mathbb{N}$ (the natural numbers) onto itself, and we set $b_k = a_{\sigma(k)}$, then $\sum_1^\infty b_k$ also equals s.

2. Let $\{(a_k; b_k)\}_1^\infty$ be a collection of intervals contained in $[0; 1]$, and suppose that

$$\sum_1^\infty (b_k - a_k) > 1.$$

Show that the set of intervals $\{(a_k; b_k)\}$ cannot be pairwise disjoint.

3. Let $(X; \mathcal{M}; \mu)$ be a measure space for which $0 < \mu(X) < \infty$, and suppose that $f \in L^\infty(X; \mathcal{M}; \mu)$. Show that $f \in L^p(X; \mathcal{M}; \mu)$ for all $p < \infty$, and also show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

4. Let $f : [0; 1] \mapsto \mathbb{R}$ be continuous (with respect to the usual, absolute-value metric) and one-to-one, and suppose that $f(0) < f(1)$. Show that f is strictly increasing on $[0; 1]$; i.e., show that, for all x and y in $[0; 1]$, $x < y$ implies $f(x) < f(y)$.

5. Show that, if $(X; \mathcal{M}; \mu)$ is any measure space, the following two statements, regarding sequences of sets $\{E_k\} \subset \mathcal{M}$, are *equivalent*: a) For all $A \in \mathcal{M}$ such that $\mu(A) < \infty$,

$$\lim_{k \to \infty} \mu(A \setminus E_k) = 0;$$

b) For all $f \in L^1(X; \mathcal{M}; \mu)$, $f\chi_{E_k} \to f$ in L^1 as $k \to \infty$. (Hint: Begin by showing that, if a) holds, b) is true for all integrable simple functions f.)

6. Let $(M; d)$ be a metric space. Show that if $\{x_k\} \subset M$ is any Cauchy sequence, and some subsequence $\{x_{n_k}\}$ converges to a point $p \in M$, then the whole sequence $\{x_k\}$ converges to p.

7. If $E \subset \mathbb{R}^d$, a point $x \in \mathbb{R}^d$ is called a *condensation point* of E if, for all $r > 0$, $B(x; r) \cap E$ is uncountable, where $B(x; r) \equiv \{y \in \mathbb{R}^d : \|x - y\| < r\}$ and $\|\cdot\|$ is the usual Euclidean norm. (Notice that x need not belong to E.) Show that, for any $E \subset \mathbb{R}^d$, the set of E's condensations points is closed.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose that $\{f_n\}$ is a sequence from $L^+(X, \mathcal{M}, \mu)$, the family of non-negative measurable functions. Show that, if $f_n \to f \in L^+$ pointwise, and

$$\lim_{n \to \infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x) < \infty,$$

then, for all $E \in \mathcal{M}$,

$$\lim_{n \to \infty} \int_E f_n(x)\, d\mu(x) = \int_E f(x)\, d\mu(x).$$

You may use standard limit theorems (Fatou, Monotone Convergence, etc.) without proof.

9. Consider the two surfaces in $\mathbb{R}^3$:

$$\Sigma_1 \equiv \{(x, y, z) : x \sin z - y \cos z = 0\};$$
$$\Sigma_2 \equiv \{(x, y, z) : x^2 + 4y^2 = 1\},$$

and let $\Gamma = \Sigma_1 \cap \Sigma_2$. Show that, for every $(x_0, y_0, z_0) \in \Gamma$, the Implicit Function Theorem implies the exisence of a differentiable, one-to-one $\phi(t) = (\phi_1(t), \phi_2(t))$, defined on some non-trivial open interval $I = (z_0 - \delta, z_0 + \delta)$, and mapping into $\mathbb{R}^2$, satisfying $\phi(z_0) = (x_0, y_0)$ and such that $(\phi_1(t), \phi_2(t), t) \in \Gamma$ for all $t \in I$.

10. Let $\mathcal{M}$ be the σ-algebra of subsets of $\mathbb{R}$ that are countable or have countable complements. Define $\mu : \mathcal{M} \mapsto \{0, 1\}$ by:

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable;} \\ 1 & \text{if } \mathbb{R} \setminus E \text{ is countable.} \end{cases} \tag{1}$$

Let $\mu^* : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty]$ be the outer measure induced by μ. a) Find an expression for μ^*, analogous to (1), valid for all $E \subset \mathbb{R}$. b) Let $\mathcal{M}^*$ be the σ-algebra of μ^*-measurable sets. Show that $\mathcal{M}^* = \mathcal{M}$.