REAL ANALYSIS PHD QUALIFYING EXAM

September 20, 2007

A passing grade is 6 problems done completely correctly, or 5 done completely correctly with substantial progress on 2 others.

1. Let $(X; d)$ be a compact metric space, where we take \compact" to mean \every open cover of X has a ¯nite subcover." Show that every sequence $\{x_n\}_1^\infty$ in X has a subsequence converging to some $z \in X$.

2a). Let $\{b_n\}$ be a sequence of positive numbers which is *unbounded*: $\sup_n b_n = \infty$. Show that there is a sequence of positive numbers $\{a_n\}$ such that $\sum a_n < \infty$, but for which $\sum a_n b_n = \infty$.
2b) Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n = \infty$. Show that there is a sequence of positive numbers $\{d_n\}$ such that $\lim_n d_n = 0$, but for which $\sum c_n d_n = \infty$.

3. Prove the following: If f is di®erentiable on $(0; 1)$ and $f'(1=4) < 0 < f'(3=4)$, there is a $c \in (1=4; 3=4)$ such that $f'(c) = 0$.

4. Let $\{f_n\}$ be a sequence of functions in $L^p(\mathrm{R}; \mathcal{L}; m)$, where $1 < p < \infty$, $\mathcal{L}$ is the Lebesgue measurable sets, and m denotes Lebesgue measure. Suppose that

$$\sup_n \|f_n\|_p < \infty: \qquad (1)$$

Show that $\{f_n\}$ is *uniformly integrable*, which means: for every ² > 0 there is a ± > 0 such that, for all $E \in \mathcal{L}$, $m(E) <$ ± implies

$$\sup_n \int_E |f_n|\, dm < ²:$$

Also, give an example of an

7. Let $(X;\mathcal{M};\,^{1})$ be a measure space, and suppose that $\{E_n\}_1^\infty$ is a sequence from $\mathcal{M}$ with the property that

$$\lim_{n\to\infty} {}^{1}(X \setminus E_n) = 07.35\text{-}1.49\text{TDerty that}$$