REAL ANALYSIS PH.D. QUALIFYING EXAM

January 31, 2009

A passing paper consists of 7 problems solved completely, or 6 solved completely with substantial progress on 2 others.

1. Let (X, d) be a metric space. A set $E \subseteq X$ is called *discrete* if there is $\delta > 0$ such that, for all x and y in E with $x \neq y$ we have $d(x, y) > \delta$. Show that a discrete set is necessarily closed. (Use any standard definition of "closed set" in a metric space.)

2. Suppose that $f : (0, 1) \to \mathbb{R}$ is differentiable on all of $(0, 1)$ and $f'(1/4) < 0 < f'(3/4)$. Show that there is a $c \in (1/4, 3/4)$ such that $f'(c) = 0$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on all of $\mathbb{R}$ and $\lim_{x \to \infty} f'(x) = A$, where A is a real number. Show that $\lim_{x \to \infty} \frac{f(x)}{}$

(a) Show that $F \in \mathcal{M}$, i.e., F is a measurable set.

(b) Prove that $\mu(F) \geq 1/100$.

(c) Give an example to show that conclusion (b) can fail if $\mu(X) = \infty$.

8. Find the value of

$$\lim_{n \to \infty} \int_0^\infty \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x}\, dx;$$

and justify your assertion by quoting appropriate facts from calculus and one or more limit theorems from measure theory.

9. Let $(X, \|\cdot\|)$ be a normed linear space.

(a) State what it means for $(X, \|\cdot\|)$ to be a Banach space, and give an example, with details, of a normed linear space that is **not** a Banach space.

(b) Let $\{x_k\}_{k=1}^\infty$ be a sequence in X and let

$$S_N = \sum_{k=1}^{N} x_k$$

be the usual N^{th} partial sum of the series $\sum_{k=1}^{\infty} x_k$. The series is said to be *summable* if the sequence $\{S_N\}_{N=1}^\infty$ of partial sums converges to an element of X. The series is called *absolutely summable* if if $\sum_{k=1}^{\infty} \|x_k\| < \infty$.

Prove that $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely summable series is summable. (You may use without proof the fact that if a Cauchy sequence has a subsequence that converges to L, then the entire sequence also converges to L.)

10. Let $\phi \in L^\infty(\mathbb{R})$ (the measure on $\mathbb{R}$ is the usual Lebesgue measure). Show that

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1 + x^2}\, dx \right)^{1/n}$$

exists and equals $\|\phi\|_\infty$.