January 10, 2008

Three hours

There are 11 questions. A passing paper consists of 6 questions done completely correctly, or 5 questions done correctly with substantial progress on 2 others.

Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Assume that every convergent subsequence converges to the same real number. Prove that there is a real number L such that the entire sequence converges to L.
[Note: The hypotheses allow the possibility that $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequences, so your proof must subsume this case.]

Let f be a real valued function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Assume $f(0) = 0$ and $|f'(x)| \le |f(x)|$ for all $x \in (0,1)$. Prove that $f = 0$ on $[0,1]$.

Let $f_n(x) = \dfrac{1}{1 + x^{2n}}$ for $x \in \mathbb{R}$.

Find where $\{f_n\}_{n=1}^{\infty}$ converges pointwise, and describe the limit function $F(x)$.

(b) Describe the intervals in $\mathbb{R}$ on which $\{f_n\}_{n=1}^{\infty}$ converges uniformly to F.

Suppose f, g and h are bounded real valued functions on $[0,1]$ with $f \le g \le h$. If f and h are Riemann integrable with $\int_0^1 f = \int_0^1 h$, prove that g is Riemann integrable.

Let $F(x,y,z) = (x^2 + z^2 - 4)^2 + xy - 2008$. Find all points (x,y,z) such that the Implicit Function Theorem does not provide a local implicit function $f(x,y) = z$ such that $F(x,y,f(x,y)) = 0$. Describe this set geometrically as a subset of $\mathbb{R}^3$ (eg., a sphere, cone, etc.).

Let $g : (0,\infty) \to \mathbb{R}$ by

$$g(x) = n \quad \text{when} \quad x \in ((n-1)^2, n^2], \qquad \text{for each } n \in \mathbb{N}.$$

Since g is increasing and left continuous let μ be the Lebesgue-Stieltjes measure obtained from g on the semiring of half open intervals:

$$\mu([a,b)) = g(b) - g(a), \qquad \text{for all } b \ge a > 0.$$

Prove that every subset of $(0,\infty)$

Let λ denote Lebesgue measure on the real line.

Prove that there is an open set **O** that is dense in $\mathbb{R}$ with $\lambda(\mathbf{O}) < 1$.

b Let **O** be any set satisfying the conclusion to part (a). Prove that $\mathbb{R}$ **O** is uncountable.

Let **O** be any set satisfying the conclusion to part (a). Prove that $\mathbb{R}$ **O** is not compact.