Numerical Analysis PhD Qualifying Exam
University of Vermont, Spring 2010

Instructions: *Four* problems must be completed, and *one*

4. **(a)**: Determine the coe□cients of an implicit, one-step, ODE method of the form

$$x(t+h) = ax(t) + bx'(t) + cx'(t+h)$$

so that it is exact for polynomials of as high a degree as possible. Begin by letting LHS = $x(h)$ and RHS = $ax(0) + bx'(0) + cx'(h)$ and fill in the missing entries in the table below. The first row and column have been filled in for you.

$x(t)$	$x'(t)$	LHS	RHS
1	0	1	a
t			
t^2			
...	...	...	...

(b): Once you have obtained the coe□cients a, b, c in part **(a)**, use Taylor Series to find the order of the local truncation error term.

5. The Dahlquist method

$$Y_{n+1} - 2Y_n + Y_{n-1} = \frac{h^2}{4}(f_{n+1} + 2f_n + f_{n-1}), \qquad \text{where} \quad f_n \equiv f(x_n, Y_n), \text{ etc.} \tag{1}$$

can be used to solve the initial-value problem

$$y'' = f(x, y), \qquad y(x_0) = y_0, \quad y'(x_0) = y'_0. \tag{2}$$

(a) Show that method (1) has the global error of order 2 when applied to (2).

(b) Show that this method is stable for any value of $h\omega$ when applied to the oscillator equation

$$y'' = -\omega^2 y, \qquad \omega > 0. \tag{3}$$

6. (a) Propose a 2nd-order accurate discretization of the equation

7. Consider a unidirectional wave equation

$$u_t = c\,u_x, \qquad -\infty < x < \infty \tag{1}$$

where c =const.

(a) Use the von Neumann analysis to determine under what condition on the ratio

$$\mu = \frac{c\kappa}{h}$$

the scheme

$$\frac{U_m^{n+1} - U_m^n}{\kappa} = c\,\frac{U_{m+1}^n - U_m^n}{h}, \tag{2}$$

approximating (1), is stable.

(b) Similarly, show that the scheme

$$\frac{U_m^{n+1} - U_m^n}{\kappa} = c\,\frac{U_{m+1}^n - U_{m-1}^n}{2h} \tag{3}$$

is unstable for any μ.

(c) Note that for a Fourier harmonic $u = \exp[i\beta x]$, the right-hand sides of (2) and (3) equal λu for some λ. (Of course, this λ is different for (2) and (3).) Use this fact to interpret your results in parts (a) and (b) in light of the stability of a certain numerical method for ODEs.