DIFFERENTIAL EQUATIONS

Qualifying Examination

January 10, 2008

INSTRUCTIONS: *<u>Two</u> problems from <u>each</u> Section must be completed, and <u>one</u> additional problem from <u>each</u> Section must be attempted. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them.* **Numeric criteria for passing***: A problem is considered completed (attempted) if a grade for it is $\geq$ 85% ($\geq$ 60%).*

Time allowed: 3 hours.

Section 1

<u>Problem 1</u>

(a) Convert the following 2nd-order IVP into a 1st-order system and solve explicitly:

$$\ddot{x} - 9\dot{x} - 10x = 0 \tag{1}$$
$$x(0) = 1 \tag{2}$$
$$\dot{x}(0) = 0 \tag{3}$$

(b) Draw the phase portrait associated with (1) and determine the stability of the fixed point at the origin. Be sure to preserve proportions in his phase portrait.

<u>Problem 2</u>

The ω-limit set of a trajectory $\gamma(t)$ is the set of points p such that there exists a sequence $t_n \to \infty$ with

$$\lim_{n \to \infty} \gamma(t_n) = p \tag{1}$$

Figure 1: Example phase portrait of a 2-D system of ODE's having a trajectory $\gamma(t)$ (dashed curve) with an ω-limit set consisting of a single limit cycle (solid curve). The fixed point in the figure is unstable (repelling).

Assuming all orbits are bounded, sketch the phase portrait of a 2-D system of ODE's having a trajectory $\gamma(t)$ with an ω-limit set consisting of:

(a) a single limit cycle and a single fixed point;

(b) two limit cycles and a single fixed point;

(c) two limit cycles and two fixed points.

Hint: A limit cycle may join two fixed points (heteroclinic), join a fixed point to itself (homoclinic), or contain no fixed points (as in Figure 1 above).

<u>Problem 3</u>

Solve the following inhomogeneous IVP explicitly:

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

Problem 4

Given the 1-D ODE

$$\dot{x} = \frac{\sin(x)}{x} - \mu; \tag{1}$$

(a) Classify the stability of all fixed points for $\mu = 0$;

(b) Draw a bifurcation diagram for (1);

(c) Find *all* values of μ for which there exactly 2 fixed points.

Section 2

Problem 5

(a) Prove the Convolution theorem. Namely, let $f(s)$ and $g(s)$ be square integrable on the infinite line, so that their Fourier transforms $F[f](\omega)$ and $F[g](\omega)$ exist, where

$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega s} f(s)ds;$$

and similarly for $F[g]$. **Show that**

$$F[f * g] = F[f] \cdot F[g];$$

where

$$f * g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s - s_1)g(s_1)ds_1.$$

(b) Consider the Heat equation on the infinite line:

$$u_t = u_{xx}; \qquad u(x;0) = q(x);$$

where $q \to 0$ and $q_x \to 0$ sufficiently fast as $|x| \to \infty$. **Show that**

$$u(x;t) = \int_{-\infty}^{\infty} \phi(x - x_1;t)q(x_1)dx_1;$$

where

$$\phi(x;t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

Hint 1: Use the result of part (a).
Hint 2: You will need the value of the following integral:

$$\int_{-\infty}^{\infty} e^{-\omega^2 a + i\omega b} d\omega = \sqrt{\frac{\pi}{a}} e^{-b^2/(4a)}; \qquad a > 0.$$

Problem 6

Consider the boundary value problem (BVP)

$$\begin{aligned} &u_{xx} + u = a; \quad a = \text{const}; \\ &u(0) = 0; \quad u(\pi) = 1. \end{aligned} \tag{I}$$

(a) Find a (simple) change of variables from u to a new variable v that reduces (I) to a BVP with homogeneous boundary conditions:

$$\begin{aligned} &v_{xx} + v = f(x); \\ &v(0) = 0; \quad v(\pi) = 0. \end{aligned} \tag{II}$$

Also, obtain the explicit form of $f(x)$.

(b) Find a formal series solution of (II).

(c) What value(s) should the constant a in (I) have in order for the solution obtained in part (b) to exist?

Problem 7

Find the displacement $u(r;\theta;t)$ of a semi-circular membrane (see the figure on the left) which satisfies the following BVP:

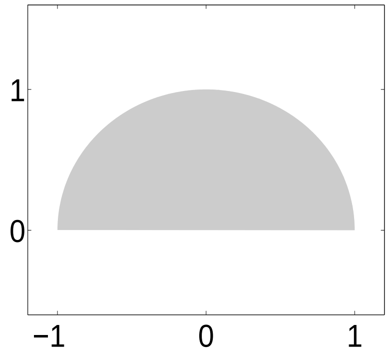

Figure 2

$$u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}; \qquad r;\theta \in M \qquad (1)$$

$$u_\theta = 0; \qquad r;\theta \in \partial M_{\text{bottom}} \qquad (2)$$

$$u = 0; \qquad r;\theta \in \partial M_{\text{top}} \qquad (3)$$

$$u(r;\theta;0) = 0; \qquad r;\theta \in M \qquad (4)$$

$$u_t(r;\theta;0) = g(r;\theta); \qquad r;\theta \in M: \qquad (5)$$

Problem 8

Consider a chain pinned at $x = L$ and such that its bottom end can move freely along the horizontal line $x = 0$. Thus, x is the vertical coordinate, and let the horizontal displacement of such a chain from the vertical line be denoted u. Separation of variables for the equation governing u leads to the following BVP:

$$xu'' + u' + \lambda^2 u = 0; \qquad x > 0 \qquad (1a)$$

$$u(L) = 0; \qquad (1b)$$

$$u(0) \text{ is bounded,} \qquad (1c)$$

and the prime denotes d/dx. In $(1a)$, λ^2 is the eigenvalue of the operator $-\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)$ subject to the boundary conditions $(1b;c)$.

(a) Verify that $u(s)$, where $s = 2\sqrt{x}$, satisfies the equation

$$\frac{d^2u}{ds^2} + \frac{1}{s}\frac{du}{ds} + \lambda^2 u = 0 \qquad (2)$$

and relate the solution $u(s)$ of (2) to a Bessel function.

(b) Put Eq. $(1a)$ (not (2)!) in the standard Sturm-Liouville form. Then derive an orthogonality relation for two eigenfunctions $u_\lambda(x)$ and $u_\mu(x)$ corresponding to *different* eigenvalues λ^2 and μ^2. Make sure to correctly determine the weight in this orthogonality relation.