COMPLEX VARIABLES PH.D. QUALIFYING EXAM

May ,

There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let f be holomorphic on a connected open set U. Prove that if $f\;z\;^{2}\quad\overline{f\;z}$ for all $z \in U$ then f is constant on U. Find all possible values for f.

2. Let be the circle of radius centered at . Evaluate with brief justification the integrals

(a) $\displaystyle\int_\gamma \frac{\overline{z}}{z -}\,dz$,

$\displaystyle\int_\gamma e^{1/z}\,dz$.

3. Find a Laurent series expansion valid in some bounded annulus centered at that contains the point z for the following function explain briefly how the inner and outer radii of the annulus are determined

$$f\;z \qquad \frac{z}{-z^2} \quad \frac{}{z - \quad ^2}:$$

4. Use the calculus of residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos\ x}{x^2}\,dx:$$

5. Prove that if f is entire and there are positive real numbers A, B and k such that $|f\;z\;| \le A \quad B|z^k|$ for all $z \in \mathbb{C}$, then f is a polynomial.

6. Let f be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume $|f\;z\;| <$ on its boundary. Prove that there is one and only one point $z_0 \in \mathbb{D}$ such that $f\;z_0 \qquad z_0$.

7. (a) Exhib

8. Define $f \; z \quad \int_0^1 \frac{dt}{tz}$

(a) Show by using Morera's Theorem that f is analytic on the open unit disc $\mathbb{D}$.

(b) Find a power series expansion for $f \; z$ valid on $\mathbb{D}$.

9. Let $P \; z$ and $Q \; z$ be polynomials with degree $Q \geq$ degree P . Prove that

$$\sum_{z_i} \mathrm{Res}_{z=z_i} \frac{P \; z}{Q \; z}$$

where the sum is over all poles z_i in $\mathbb{C}$ of the rational function $\frac{P}{Q}$.

10. Suppose f is analytic on the punctured unit disc $\mathbb{D} - \{ \; \}$ and the real part of f is positive there. Prove that f has a removable singularity at .