ALGEBRA H D QUALIFYING EXAM

September 27, 2008

A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.

Section A

In this section you may quote without proof basic theorems and classifications from group theory and group actions as long as you state clearly what facts you are using.

Let p and q be distinct primes and let G be a group of order p^3q.

Show that if $p > q$ then a Sylow p-subgroup of G is normal in G.

Assume G has more than one Sylow p-subgroup. Show that if the intersection of any pair of distinct Sylow p-subgroups is the identity, then G has a normal Sylow q-subgroup.

c **Assume the Sylow p-subgroups of G are abelian. Show that G is not a simple group. (Do not quote Burnside's p^aq^b-Theorem.)**

Let G be a finite group acting transitively (on the left) on a nonempty set Ω. For $\omega \in \Omega$ let G_ω be the usual stabilizer of the point ω:

$$G_\omega = \{g \in G \mid g\omega = \omega\}$$

where $g\omega$ denotes the action of the group element g on the point ω.

Prove that $hG_\omega h^{-1} = G_{h\omega}$, for every $h \in G$.

Assume G is *abelian*. Let N be the kernel of the transitive action. Prove that $N = G_\omega$ for every $\omega \in \Omega$, and deduce that $|G : N| = |\Omega|$.

Let N be a normal subgroup of the group G and for each $g \in G$ let φ_g denote conjugation by g acting on N, i.e.,

$$\varphi_g(x) = gxg^{-1} \quad \text{for all } x \in N.$$

Prove that φ_g is an automorphism of N for each $g \in G$.

Prove that the map $\Phi : g \mapsto \varphi_g$ is a homomorphism from G into Aut(N), where Aut(N) is the automorphism group of N.

c **Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of Aut(N).**

Section B

Let X be any nonempty set and let R be the (commutative) ring of all integer-valued functions on X under the usual pointwise operations of addition and multiplication of functions: $R = \{f \mid f : X \to \mathbb{Z}\}$. For each $a \in X$ define $M_a = \{f \in R \mid f(a) = 0\}$.

Prove that M_a is a prime ideal in R.

Prove that M_a is not a maximal ideal in R.

c **Find all units in R.**

d **Find all zero divisors in R.**

Let F and K be nite elds with $F \quad K$. Let $F[x]$ and $K[x]$ denote the respective polynomial rings in the variable x, so $F[x]$ is a subring of $K[x]$.

Prove that if M is any maximal ideal in $K[x]$, then M \ $F[x]$ is a maximal ideal in $F[x]$.

Give an explicit example of commutative rings $A \quad B$ and a maximal ideal I of B such that I \ A is not a maximal ideal of A.

Let R be a Principal Ideal Domain, let p and q be distinct primes in R, and let $a = p \ q$ for some 2 $\mathbb{Z}^+$. Let M be any R-module annihilated by (a). Prove that

$$M = M$$