ALGEBRA PH.D[redacted]NG EXAM

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A passing paper consists of five problems solved [redacted] or four solved completely plus signif[redacted] progress on two other problems; in both cases [redacted] problems solved completely must inc[redacted] one from each of Sections A, B and C.

Sectio[redacted]

In this section you may quote without proof basic [redacted] classifications from gro[redacted] as long as you state clearly what facts you are using.

1. Let G be a group of order 2457 (note that $2457 = 3^3 \cdot$ [redacted]

   (a) Compute the number, n_p, of,,

5. Let R be a Principal Ideal Domain with field of fractions F and assume $R = F$. As usual we may view F as a module over its subring R.

   (a) Prove that every finitely generated R-submodule of F is a cyclic R-module.

   (b) Deduce from (a) that F cannot be a finitely generated R-module.

   (You may quote results about modules over PIDs.)

6. Let $\mathbb{F}_q$ be the finite field with q elements. Find the number of similarity classes of 5×5 matrices A over F_q that satisfy $A^q = I$, where I is the identity matrix. (Justify your answer. You do not need to exhibit representatives of the classes.)

Section C.

7. Let $f(x) = x^6 - 6x^3 + 1$ and let α, β be the two real roots of $f(x)$ with $\alpha > \beta$. You may assume $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ in $\mathbb{C}$.

   (a) Exhibit all six roots of $f(x)$ in terms of radicals involving only integers and powers of ω, where ω is a primitive cube root of unity.

   (b) Prove that $K = \mathbb{Q}(\alpha, \omega)$ and deduce that $[K : \mathbb{Q}] = 12$. [Hint: What is $\alpha\beta$?]

   (c) Prove that $G = \mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup N such that G/N is the Klein group of order four.

8. Let n be a given positive integer and let E_{2^n} be the elementary abelian group of order 2^n (the direct product of n copies of the cyclic group of order 2). Show that there is some positive integer N such that the cyclotomic field $\mathbb{Q}(\zeta_N)$ contains a subfield F that is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(F/\mathbb{Q}) = E_{2^n}$, where ζ_N is a primitive N^{th} root of 1 in $\mathbb{C}$.

9. Let p be a prime and let $q = p^n$ for some $n \quad \mathbb{Z}^+$.

   (a) What is the degree of the extension $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$? Describe how the Frobenius automorphism, σ, for this extension acts on the elements of $\mathbb{F}_{q^2}$.

   (b) Define the norm map

   $$N : \mathbb{F}_{q^2}^{\times} \longrightarrow \mathbb{F}_q^{\times} \qquad \text{by} \qquad N(a) = a\sigma(a).$$

   Prove that this norm map is surjective. [Hint: Note that N is a homomorphism of multiplicative groups. Use (a) and facts about finite fields to find the order of its kernel.]